DIAS-STP-85-22

HERMANN WEYL AND THE APPLICATION OF GROUP THEORY TO QUANTUM MECHANICS

BY

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I was especially pleased to be invited to address this Congress because Hermann Weyl's work has had such an enormous influence on my own. A large part of the latter, both on the general theory of unitary group representations, and on its applications to quantum mechanics grew out of my study of a celebrated paper by M.H. Stone (who some years earlier had been my thesis advisor). This paper in turn seems to have been directly inspired by the 1927 paper in which Weyl first sketched his own ideas on the importance of group theory in quantum mechanics. Indeed the whole purpose of Stone's paper was to give exact formulations and announce rigorous proofs of two theorems suggested by Weyl's work. Moreover when I later embarked on a serious attempt to understand quantum mechanics my most important sources were von Neumann's "Mathematische Grundlagen der Quantenmechanik" and Chapter II of the English translation of Weyl's "Gruppentheorie und Quantenmechanik". Although we had very little personal contact I must consider Hermann Weyl as one of my most important teachers.

Weyl's work on the applications of group theory to quantum mechanics was immediately preceded by important contributions to the abstract theory of group representations including his far reaching observations on the essentially group theoretical character of Fourier analysis. To see all of this work in its proper perspective it will be useful to begin with a sketch of the historical background.

During roughly the first quarter of the twentieth century three exciting new developments were being pursued in mathematics and physics which seemed to have nothing to do with one another. One was in analysis, one in algebra and one in physics. They may be described briefly as follows:

(a) The work of Hilbert on integral equations and the invention of the Lebesgue integral leading to the Riesz-Fischer theorem, Hilbert's spectral theorem and functional analysis.
(b) The work of Frobenius, Burnside and Schur in inventing and developing the representation theory of finite groups.
(c) The development of the so called "Old quantum theory" by Planck, Einstein and Bohr.

After making some remarks about each of these topics in turn I shall sketch the remarkable events of the years 1925-1927 in which the many anomalies and contradictions of the old quantum theory were removed by the invention of quantum mechanics and in which it turned out that the mathematical developments listed in (a) and (b) above were just what was needed for the proper formulation and implementation of this new and subtle refinement of classical mechanics. Moreover (a) and (b) were not just brought together by their common application to (c). It was found that Fourier analysis, spectral theory and the theory of group representations could all be regarded as special cases of one far reaching unified theory. In my opinion the work of Hermann Weyl was the single most important factor in bringing about this startling unification of these apparently quite diverse topics.

Hilbert began his work on integral equations immediately after hearing about Fredholm's 1900 work on the same subject in the winter semester 1900-1901. The Lebesgue integral was introduced in Lebesgue's thesis of 1902 and by 1907 had led to the celebrated Riesz-Fischer theorem which put Fourier analysis in a much more satisfactory and elegant form. The high point of Hilbert's work on integral equations was his celebrated spectral theorem for bounded self adjoint operators in Hilbert space which was in turn suggested by his strategy of exploiting the analogy between integral operators and the matrices of linear algebra.

Because of the far reaching role it will play in what follows we pause to explain the nature of this theorem of Hilbert's and to introduce some technical terminology. In broad terms it is an infinite dimensional generalization of the classical theorem stating that every nxn matrix $\|a_{ij}\|$ of complex numbers, which is self adjoint in the sense that $\overline{a_{ij}} = a_{ji}$, is diagonizable. This means that there exists a matrix $\|u_{ij}\|$ which is "unitary" in the sense that $\|u_{ij}\| \cdot \|\overline{u}_{ji}\| = I$ where I is the identity matrix and such that $\|u_{ij}\|^{-1} \|a_{ij}\| \|u_{ij}\| = \|a'_{ij}\|$ is a "diagonal" matrix in the sense that $a'_{ij} = 0$ when $i \neq j$. From a more geometrical point of view one thinks of $\|a_{ij}\|$ as defining a linear operator T_A in an n dimensional complex vector space with an "inner product" and then diagonizability means that there is a basis of mutually orthogonal vectors each of which is an "eigenvector" in the sense that $T_A(\phi_j) = \lambda_j \phi_j$ where λ_j is a real number. This can be restated in the form: T_A is a direct sum of "constant" operators each acting in a one dimensional subspace. Of course the operator taking any ϕ into $\lambda_j \phi$ is the constant operator in the jth subspace.

When one replaces the finite dimensional space by a complete infinite dimensional one — a so called Hilbert space the obvious generalization of this theorem is no longer true. Many important self adjoint operators have no eigenvectors except 0. To understand Hilbert's generalization it is useful to reformulate the classical theorem in what may seem like a perverse manner. For each subset E of the real line let $\lambda_{i_1}, \lambda_{i_2} \cdots \lambda_{i_r}$ be the eigenvalues of T_A which happen to lie in E and let P_E denote the unique operator such that $P_E(\phi_{j_k}) = \phi_{j_k}$ and $P_E(\phi_j) =$ when λ_j is not one of the λ_{i_s} i.e. when λ_j is not in E. Then each P_E is a self adjoint operator with the property that $P_E^2 = P_E$. Such self adjoint idempotent operators are called <u>projections</u> and P_E is in fact the projection of the whole space on the subspace spanned by the eigenvectors $\phi_{\lambda_{i_1}}, \phi_{\lambda_{i_2}} \cdots \phi_{\lambda_{i_r}}$. This projection valued set function $E \longrightarrow P_E$ is easily seen to have the following simple properties.

(1) $P_\phi = 0$ and $P_R = I$ where I is the identity operator, ϕ is the empty set and R the whole real line.

(2) $P_E P_F = P_F P_E = P_{F \cap E}$ for all subsets E and F of R.

(3) If $E_1, E_2 \ldots$ are mutually disjoint subsets of R then $P_{E_1 \cup E_2 \cup \cdots} = P_{E_1} + P_{E_2} + P_{E_3} + \ldots$.

Moreover given this set function $E \to P_E$ we can easily reconstruct all the matrix elements of T_A and hence T_A itself. Indeed for two arbitrary vectors ϕ and ψ one proves that $(T_A(\phi)\cdot\psi) = \sum_{\lambda \in R} \lambda (P_{\{\lambda\}}(\phi)\cdot\psi)$ where $\{\lambda\}$ denotes the set whose only element is λ. Of course $P_{\{\lambda\}} = 0$ except when λ_i is one of the eigenvalues of T_A and the sum on the right hand side is actually finite and equal to $\sum_{i=1}^{n} \lambda_i (P_{\{\lambda_i\}}(\phi)\cdot\psi)$ In our perverse and awkward looking reformulation the classical diagonalization theorem says that for every self adjoint operator T_A in a finite dimensional Hilbert space there exists a unique projection valued set function $E \to P_E$ having properties (1) (2) and (3) listed above such that

$$(T_A(\phi)\cdot\psi) = \sum_{\lambda \in R} \lambda \left(P_{\{\lambda\}}(\phi)\cdot\psi\right) .$$

The great advantage of this reformulation is that with minor modifications it is also true for self adjoint operators in any separable infinite dimensional Hilbert space; and in somewhat different form this is what Hilbert proved for all bounded self adjoint operators. After making a few preliminary definitions we shall give a precise statement of this form of Hilbert's theorem. We define a subset of the real line to be a Borel set if it can be built up out of open intervals by repeated application of the process of countable union, countable intersection and the taking of complements. We then define a <u>projection valued measure</u> on the line to be a function $E \to P_E$ from the Borel subsets of R to the projection operators in some separable Hilbert space H which has properties (1) (2) and (3) listed above. Of course in (2) and (3) we must restrict $E, F, E_1, E_2, \ldots$ to be Borel sets. If there exists a finite interval J = all x with $-M \leq x \leq M$ such that $P_J = I$ one says that the projection valued measure P has <u>bounded support</u>. Finally we notice that if P is any projection value on R and ϕ is any vector in the Hilbert space H of P then $E \to (P_E(\phi)\cdot\phi)$ is an ordinary numerical measure on the Borel subsets of R and it makes sense to consider $\int f(x)\, d(P_x(\phi)\cdot\phi)$ for suitably restricted complex valued functions i.e. to integrate f with respect to the measure $E \to (P_E(\phi)\cdot\phi)$. More generally one can integrate f with respect to the complex valued measure $E \to (P_E(\phi)\cdot\psi)$. With these preliminaries we may state:

<u>Hilbert's spectral theorem:</u> Let T be any bounded self adjoint operator in the separable Hilbert space H. Then there exists a unique projection valued measure with bounded support, $E \to P_E$, defined on the Borel subsets of R and whose values are projection operators in H such that for all ϕ and ψ in H we have the identity

$$\left(T(\phi)\cdot\psi\right) = \int_R x\, d(P_x(\phi)\cdot\psi) .$$

Conversely it is rather easy to show that every projection valued measure with bounded support is related in this way to a unique bounded self adjoint operator T. One finds also that T has a basis of eigenvectors as in the classical finite dimensional case if and only if the corresponding P has countable support; that is if and only if there exists a countable set $E = \{\lambda_1, \lambda_2 \cdots\}$ such that $P_E = I$. In that case the ranges of the projections $P_{\{\lambda_\nu\}}$ constitute a direct sum decomposition of the Hilbert space and in each of these T is a constant times the identify. In the general case $P_{\{\lambda\}}$ can well be zero for every real number λ and then one can think intuitively of the spectral theorem as stating that the self adjoint operator is a "direct integral" of constant operators rather than a direct sum.

Let us now turn to topic (b).
The theory of representations and their characters for finite groups was invented in 1896 by G Frobenius. He did this in a more or less deliberate attempt to generalize a much simpler notion that had been used at least implicitly in number theory since its first appearance in Gauss' celebrated "Disquisitiones Arithmeticae" published in 1801. The word character was introduced by Gauss. Let G be any finite commutative group. Then by definition a character χ of G is a complex valued function on G such that $\chi(xy) = \chi(x)\chi(y)$ for all x and y on G. Gauss considered only those characters χ such that $\chi^2 = 1$ and only those groups G which occurred in the number theory of binary quadratic forms. One of his fundamental contributions was to show that the set of all equivalence classes of forms with a given discriminant would be made into a finite commutative group in a certain way and his "characters" were designed to distinguish between inequivalent forms with the same discriminant. Dedekind generalized Gauss notion in 1878 by removing the restriction that $\chi^2 = 1$ and three years later Weber pointed out that it made sense for arbitrary commutative groups. A large part of its importance stems from the following simple theorem:

Theorem: Any complex valued function on the finite commutative group G may be written uniquely in the form $f(x) = \sum_{\chi \in \hat{G}} c_\chi \chi(x)$ where $\hat{G}$ denotes the set of all characters of G. Moreover the complex coefficients c_χ may be computed from f by the formula

$$c_\chi = \frac{1}{o(G)} \sum_{x \in G} f(x)\bar{\chi}(x)$$

where $o(G)$ is the number of elements in G.

While this theorem was not explicitly formulated until much later one can, with hindsight, recognize its use as a key element in a number of significant proofs in nineteenth century number theory.

The theory of higher reciprocity laws in number theory which was created by Gauss in 1828 and carried to a sort of conclusion by Kummer in the 1850's led more or less directly to Dedekind's theory of general algebrac number fields in 1870. The symmetry groups (Galois groups) of these fields - unlike those considered by Kummer were not always commutative and problems that Dedekind could solve in the commutative case using characters as a tool remained baffling in the non commutative case. Dedekind appealed to Frobenius for help and Frobenius responded by showing how to generalize the theory of characters from commutative to non commutative finite groups. Of course the definition of character makes sense for non commutative groups but it does not go far enough. Every finite group G has a largest normal subgroup N such G/N is commutative. The classical characters of G are all trivial on N and reduce essentially to characters of the commutative quotient group G/N.

Frobenius' solution is a little easier to explain in a second version which he found a year later. One simply replaces complex numbers by non singular nxn matrices and defines a matrix representation of the finite group G to be a matrix valued function $x \to A(x)$ defined on G such that A(xy) = A(x)A(y). When n is one we recover the classical characters but now the non commutativity of matrix multiplication makes it possible for our notion to be significant for the non commutative part of G. If $x \to A(x) = \| a_{ij}(x) \|$ is a matrix representation of G one defines the character of this representation to be $\sum_{j=1}^{n} a_{jj}(x)$. For those representations with n=1 the characters in this sense are precisely the classical characters of Gauss as generalized by Dedekind and Weber. We shall refer to these henceforth as one dimensional characters.

Just as with spectral theory it is often illuminating to think in terms of linear transformations rather than matrices and to define a representation accordingly. Since the diagonal sum $\sum_{j=1}^{n} a_{jj}$ is the same for all matrix representations of the same linear transformation there is no difficulty about defining characters just as before. A representation $x \to A_x$) of G by linear transformations in some vector space $H(A)$ is said to be irreducible if there are no proper subspaces M of H(A) such that $A_x(M) = M$ for all x. One proves that every representation is a direct sum of irreducible representations in the same sense that every self adjoint operator is a direct sum of constant operators. Defining an irreducible character to be the character of an irreducible representation one sees easily that every character is of the form $n_1\chi_1 + n_2\chi_2 + \cdots n_j\chi_j$ where $\chi_1, \chi_2 \ldots \chi_j$ are distinct irreducible characters, j and the n_j are positive integers. This decomposition is unique. One proves also that, for each finite group G, the number of distinct irreducible characters is finite — and in fact equal to the number of conjugacy classes in the group.

Unlike the commutative case in which all irreducible characters are one dimensional and quite easy to determine; finding the irreducible representations and corresponding characters of a finite non commutative group can be very difficult. In the period 1896 to 1924 it was an exciting new field of investigation. Even today there are serious and interesting unsolved problems.

Topic (c) grew out of late 19th century attempts to explain the spectrum of radiation from a so called "black body" by combining electric magnetic theory with statistical mechanics. These attempts were successful only when the temperature was large relative to the frequency of the radiation. It is significant that other predictions of statistical mechanics also failed at low temperatures. Then in 1900 Max Planck made the remarkable discovery that one could derive a formula valid for all temperatures from the bizarre assumption that the energy of a harmonic oscillator of frequency ν could not take on a continuum of values but only values of the form $nh\nu$ where h is a universal constant (now known as Planck's constant) and n = 0,1,2,3... . In 1905 and 1906 Einstein used similar ad hoc "quantization" hypotheses to explain the low temperature behaviour of specific heats and the so called photo electric effect. A bit later in 1913 H. Bohr found a similar explanation of the wave lengths occurring in the spectrum of atomic hydrogen. There was success after success but no real understanding because the various quantization hypotheses could not be reconciled with the principles of classical mechanics. For a quarter of a century physicists had to live with a level of logical incoherence which was quite strange to them.

So much for the historical background. We are now ready to talk about the contributions of Hermann Weyl. These began in 1924 with his work on extending the representation theory of finite groups to a class of infinite continuous groups: the compact Lie groups. They culminated in 1927 with Weyl's work in (a) Unifying group representation theory with Fourier analysis (b) Helping to clarify the structure of the new quantum mechanics that emerged to replace the old quantum mechanics of 1900-1924 after the fundamental discoveries of Heisenberg and Schrödinger in late 1924 and early 1925 (c) Unifying spectral theory with the theory of group representations while applying both to the new quantum mechanics.

In 1924 I. Schur a student of Frobenius and one of the leading early workers in the theory of group representations published a paper indicating how one could generalize certain features of the theory for finite groups to the orthogonal groups. The key idea was to replace summing over the group elements by integrating over the (compact) group manifold —using a definition of integration introduced earlier in another connection by Hurwitz. Weyl immediately became interested and wrote a letter to Schur indicating how one might go much further. This was soon followed by a celebrated sequence of papers "Theorie der Darstellung kontinuerlicher halb ennfacher Gruppen durch lineare Transformationen, I, II, III" published in the Mathematische Zeitschrift in 1925 and 1926. In these Weyl gave a very complete and elegant account of the irreducible representations and their characters for all of the compact semi simple Lie groups. Further details will be found in Professor Freudenthal's talk at this Congress as well as below where we will indicate certain connections with Weyl's work on quantum mechanics.

A year later in 1927 Weyl published a further paper on the representation theory of compact Lie groups; this one written in collaboration with his student F. Peter. It was entitled "Die Vollstandigkeit der primitiven Darstellungen eine geschlossener kontinuerlich Gruppe" and contains the now celebrated Peter-Weyl theorem. This theorem can be stated in several ways but can perhaps best be understood here as a new and unexpected generalization of the fact that quite general periodic functions on the real line may be expanded in Fourier series $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$; that is that the complex trigonemetric functions $e^{2\pi i N x}$ form a "complete" set of (orthogonal) function amongst functions of period 2π. Observe now that the set of all real numbers is a continuous group under addition and that the "quotient group" T obtained by identifying numbers that differ by an integer multiple of 2π is a compact continuous group. Observe also that a complex valued function of period 2π may be regarded as a complex valued function on T and that in particular the functions $x \to e^{inx}$ $n = 0, \pm 1, \pm 2, \cdots$ are characters of T; indeed one can show that every continuous character of T is of this form. Thus one can restate the completeness of the complex trigonometric functions $x \to e^{inx}$ as the completeness of the continuous characters on a certain compact commutative Lie group. The idea of Peter and Weyl was that there should be an analogous result for any compact Lie group; commutative or not with continuous irreducible representations replacing continuous characters. More precisely one replaces characters by matrix elements of irreducible representations. The proof of Peter and Weyl made use of the theory of integral equations and yielded a new proof in the classical Fourier case. It is interesting that their theorem can also be interpreted in purely group representational terms where it asserts that the so called regular representation of the group (which for non finite groups is infinite dimensional) may be decomposed as a direct sum of irreducible representations and every irreducible representation appears with a multiplicity equal to its dimension.

The corresponding result for finite groups is much easier and was one of the earliest theorems of Frobenius.

From the point of view of the general structure of mathematics the Peter Weyl theorem is especially interesting in that it unifies Fourier analysis with the theory of group representations and at the same time points out and underlines the essentially group theoretic nature of Fourier analysis. It is well known that Fourier analysis has been a well nigh indispensible tool in mathematical physics since its introduction by Fourier in 1807 — above all because of its power in solving linear partial differential equations with constant coefficients. Moreover as I have indicated in part (b) of my remarks on the historical background for Weyl's work, the characters of finite commutative groups have played an almost equally important role in the nineteenth century development of number theory — especially through the use of the formula $f(x) = \sum_{\chi \in \hat{G}} c_\chi \chi(x)$ where $c_\chi = \frac{1}{o(G)} \sum_{x \in G} f(x) \overline{\chi(x)}$ If the reader will compare these formulae with the formulae $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ where $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$ he or she will have no difficulty in seeing that both pairs are special cases of one general assertion about compact commutative Lie groups. In other words the paper of Peter and Weyl can be regarded as showing that one of the most important methods of nineteenth mathematical physics is in essence the same as one of the most important methods in nineteenth century number theory. If the reader knows of any earlier recognition of this important fact the author will be most interested in hearing about it.

Of course the unification just described is only part of the story. The Peter-Weyl theorem applies not only to commutative compact Lie groups but to compact Lie groups in general. We have before us not only a <u>unification</u> of the Fourier analysis of periodic functions on the line with its analogue for functions on finite commutative groups but a natural <u>generalization</u> in which the group T and the finite commutative groups are replaced by arbitrary compact Lie groups. (We are here extending the definition of Lie groups to include all topological groups in which the connected component of the unit element is a Lie group in the restricted sense. In particular all discrete groups are included). This suggests the possibility that this extended non commutative theory might have applications even more far reaching than the nineteenth century applications of the commutative theory to both physics and number theory. In the ensuing half century this possibility has become an important reality the full extent of which is far from widely appreciated.

The applications to physics began in 1927, the very year in which the Peter-Weyl theorem was published but before describing Weyl's remarkable contributions to this development we must go back to 1924 and pick up another thread; the development of quantum mechanics out of the "old quantum theory" which started in late 1924 and early 1925 with remarkable discoveries of W. Heisenberg and E. Schrödinger respectively. In what seemed to be quite different ways each had managed to derive the discrete energy levels of the hydrogen atom without making a priori discreteness assumptions as Bohr had done. The immediately ensuing developments took place so rapidly that it is difficult to know what happened. Much communication took place by word of mouth and private letters and one cannot trace the course of events through an orderly sequence of publications. Let it suffice to say that many individuals were involved including not only Heisenberg and Schrödinger but also Born, Bohr, Pauli, Jordan and especially Dirac. By 1927 physicists had at their disposal a subtle refinement of classical mechanics which (a) was internally consistent, (b) automatically implied the mysterious "quantization rules" of the old quantum theory, (c) reduced to classical mechanics in the limiting case of large masses and energies and (d) explained much that was beyond the reach of both classical mechanics and the old quantum theory. A key feature of this new mechanics was its abandonment of the idea that the future of a system of particles was determined by the values of the coordinates of the particles and their rates of change at some particular time t_0. Indeed it was decided that exact simultaneous values for all coordinates and velocities could not be determined, or rather did not in principle exist. It did make sense however to assign simultaneous probability distributions to all dynamical variables (observables) and the aim of the theory became that of studying how these probability distribution changed with time — and also which such distributions could exist simultaneously.

The answers to these questions about changing probabilities, as formulated by the physicists, left something to be desired both in unity and in mathematical precision. These deficiencies were removed by J. von Neumann in an extremely influential paper published in 1927. von Neumann observed that Hilbert's spectral theorem, suitably generalized to unbounded self adjoint operators was just the tool that was needed. In his formulation the set of all observables in a quantum mechanical system are in a definite one-to-one correspondence with the (not necessarily bounded) self adjoint operators in a separable infinite dimensional Hilbert space H and the possible states (possible simultaneous probability distributions) similarly correspond one to one to the unit vectors in H; except that ϕ and $e^{i\lambda}\phi$ correspond to the <u>same</u> state whenever λ is real. The significance of this correspondence is as follows. If A is the self adjoint operator corresponding to some observable $\mathcal{O}$ and ϕ is a unit vector defining a state s then the probability that $\mathcal{O}$ will be found to have a value in the set E of real numbers when measured in the state s will be $(P_E^A(\phi)\cdot\phi)$ Here $E \to P_E^A$ is the projection valued measure canonically associated to A by the generalized spectral theorem. As noted above the function $E \to (P_E^A(\phi)\cdot\phi)$ is a measure and when $\|\phi\| = 1$ it is a probability measure. Studying how these probability measures change with time reduces to studying how the state vectors ϕ change with time and this is accomplished by integrating a differential equation of the form $\frac{d\phi}{dt} = i\, H(\phi)$ where H is a suitable self adjoint operator.

At very nearly the same time von Neumann pointed out to Wigner that an analysis the latter had recently made in connection with the applications of quantum mechanics to the understanding of atomic spectra could be clarified and extended by using the representation theory of the group S_n of all permutations of n objects. In the applications n is the number of electrons in the atom in question and Wigner had managed to deal with the case n=3 by elementary methods. In taking up von Neumann's suggestion and developing his own method accordingly, Wigner became the pioneer in applying the representation theory of finite groups to the new quantum mechanics. Soon thereafter he discovered how to clarify the classification of spectral terms by the angular momenta of the atomic states by using the representation theory of the rotation group in three dimensions — a non commutative compact Lie group.

At this point we are ready to return to the work of Hermann Weyl. In that magic year 1927, Weyl published a paper entitled "Quantenmechanik und Gruppentheorie" in which he applied the theory of group representations to quantum mechanics in a rather different way than Wigner and at the same time contributed in a significant way to von Neumann's clarification of the conceptual foundations of this new mechanics. While so doing he indicated that Hilbert's spectral theorem could be regarded as a theorem about the unitary representation theory of a certain <u>non compact</u> connected Lie group — the additive group of the real line; thus pointing the way to encompassing the spectral theory of self adjoint operators as a special case of an enlarged theory of group representations.

In the introduction to his paper Weyl begins with the statement that one can distinguish sharply between two questions in (the foundations of) quantum mechanics.

(1) How does one arrive at the self adjoint operators which correspond to various concrete physical observables? (2) What is the physical significance of these operators; i.e. how does one deduce physical statements? He goes on to say that question (2) has been satisfactorily answered by von Neumann (in the paper we have just discussed) but that

von Neumann's treatment can be supplemented in certain ways. Coming back to question (1) he asserts that it is a deeper question which has not yet been satisfactorily treated and that he proposes to do so with the help of group theory. "Hier glaube ich mit Hilfe der Gruppentheorie zu einer tieferen Einsicht in den wahren Sachverhalt gelangt zu sein". This may be translated as "Here with the help of group theory I believe I have succeeded in arriving at a deeper insight into the true nature of things". In a footnote he cites the work of Wigner and says (Translation) "this connection with group theory lies in quite a different direction than the researches of Mr. Wigner who .....".

The part of Weyl's article following the introduction is divided into three parts of which part II will be our principal concern. In part I Weyl introduces the fundamental distinction between mixed and pure states. von Neumann found this independently, but did not publish it in the paper cited above. Weyl acknowledges the overlap in a footnote added in proof. The concept of mixed state, which is fundamental for quantum statistical mechanics is usually mistakenly attributed to von Neumann alone. For example one often speaks of the "von Neumann density matrix". In part II he addresses the first of the two questions raised in the introduction. His particular concern is to find some a priori justification for the fact that the self adjoint operators $Q_1, Q_2 \ldots Q_n$ $P_1, P_2 \ldots P_n$ which correspond to position coordinates and momentum components respectively should satisfy the now celebrated Heisenberg commutation relations $P_j Q_j - Q_j P_j = \frac{h}{i}$ (where h is Planck's constant) with all other pairs commuting. To discuss this problem he makes use of a fundamental connection between self adjoint operators and continuous unitary representations of R; the additive group of the real line. Indeed if A is a finite dimensional self adjoint operator one can make sense of e^{iAt} in several ways; in particular by diagonalizing A and replacing each eigenvalue λ_j by $e^{it\lambda_j}$. One sees easily that $e^{i(t_1+t_2)A} = e^{it_1A}e^{it_2A}$ so that e^{itA} is such a unitary representation and it is not hard to show conversely

N.B. e and e should be considered the <u>same</u>

that for every continuous unitary representation $t \to U_t$ of R there is a unique self adjoint operator A such that $U_t = e^{iAt}$. In fact $A = \frac{1}{i}\frac{d}{dt}U_t\big]_{t=0}$ Weyl suggests that by using Hilbert's spectral theorem one can probably extend this correspondence to the infinite dimensional case with unbounded self adjoint operators included. If so one can replace the Q_i and P_j by unitary representations U^i and V^j where $U_t^i = e^{iQ_i t}$ and $V_s^j = e^{iP_j s}$ and attempt to rephrase the question in terms of commutation relations for the U_t^i and V_s^j. This is easily done and the answer is that $PQ - QP = \frac{h}{i}$ if and only if $e^{isP}e^{itQ} = e^{\frac{ist}{h}}e^{itQ}e^{isP}$ for all real numbers s and t. It is of course more or less obvious that two self adjoint operators A and B will actually commute with one another if and only if e^{iAt} and e^{iBs} commute with one another for all t and s. Thus the Heisenberg commutation relations for a system with n particle coordinates may be rewritten in the form

$$e^{isP_k}e^{itQ_j} = e^{itQ_j}e^{isP_k} \quad \text{WHEN } j \neq k$$

$$= e^{\frac{ist}{h}}e^{itQ_j}e^{isP_k} \quad \text{WHEN } j = k$$

$$e^{isP_k}e^{itP_j} - e^{itP_j}e^{isP_k} = e^{isQ_k}e^{itQ_j} - e^{itQ_j}e^{isQ_k} = 0$$

for all j and k.

Notice now that if we define

$$U_{t_1, t_2 \ldots t_n} = e^{i(t_1P_1 + t_2P_2 + \ldots t_nP_n)} = e^{it_1P_1}e^{it_2P_2} \cdot e^{it_nP_n}$$

which we may do whenever the P_j commute with one another then

$$U_{(t_1,t_2 \ldots t_n)+(t_1',t_2' \ldots t_n')} = U_{t_1,t_2 \ldots t_n} U_{t_1',t_2' \ldots t_n'}$$

so that $t_1, t_2 \ldots t_n \to U_{t_1, t_2 \ldots t_n}$ is a unitary representation of the <u>commutative group</u> R^n of all n tuples of real numbers under addition. Similarly $s_1 s_2 \ldots s_n \to V_{s_1, s_2 \ldots s_n} = e^{i(s_1Q_1 + s_2Q_2 + \ldots s_nQ_n)}$

is also a unitary representation of the commutative group in question. Moreover assuming the truth of Weyl's conjecture about the general correspondence between self adjoint operators and unitary representations of R the additive group of the real line one shows easily that every (continuous) unitary representation of R^n may be written uniquely in the form $x_1, x_2, \ldots x_n \rightarrow e^{i(x_1 A_1 + \cdots x_n A_n)}$ where the A_j are mutually commuting self adjoint operators. In these terms as Weyl observed one may restate the Heisenberg commutation rules in the following terms. The Q_j and the P_j commute among themselves and the unitary representations U and V of R which this makes possible satisfy

$$U_{t_1, t_2, \ldots t_n} V_{s_1, s_2, s_3, \ldots s_n} = V_{s_1 s_2 \ldots s_n} U_{t_1, t_2, \ldots t_n} e^{i(s_1 t_1 + s_2 t_2 \ldots s_n t_n)} .$$

These are the Heisenberg commutation relations in "integrated" or "Weyl" form.

Let us observe next (with Weyl) that if we define $W_{t_1, t_2, \ldots t_n, s_1, s_2 \ldots s_n}$ to be the operator $U_{t_1, t_2, t_3, \ldots t_n} V_{s_1 s_2 \ldots s_n}$ then $t_1, t_2 \ldots t_n, s_1, s_2 \ldots s_n \longrightarrow W_{t_1 t_2, \ldots t_n, s_1 s_2 \ldots s_n}$ is not a unitary representation of R^{2n}. The factor $e^{\frac{i}{h}(t_1 s_1 + \cdots t_n s_n)}$ interferes. However it is a so called projective or ray representation. Quite generally if G is a group and t $\longrightarrow L_t$ is a linear operator valued function on G one says that L is a projective (or ray) representation if $L_{st} = \sigma(s,t) L_s L_t$ where $\sigma(s,t)$ is a complex number depending on s and t. When $\sigma(s,t) \equiv 1$ the definition reduces to that of group representation as given earlier. For finite groups projective representations were studied in some depth by I. Schur in papers published in 1904 and 1907. Weyl points out that projective representations are especially relevant in quantum mechanics because whenever V^1 and V^2 are unitary operators in a Hilbert space H and $V^2 = e^{i\theta} V^1$ where θ is a real number then V^1 and V^2 define exactly the same permutation of the pure states of the system. It follows that in dealing with groups of symmetries each unitary operator defining a symmetry is determined only up to a "phase factor" $e^{i\theta}$ and so the identity $L_{st} = L_s L_t$ must be relaxed to read $L_{st} = \sigma(s,t) L_s L_t$ where $\sigma(s,t)$ is a complex number of modulus one.

Weyl observed not only that the Heisenberg commutation relations are equivalent to the statement that W is a projective unitary representation of the commutative group R^{2n} with respect to the "multiplier" σ where $\sigma(t_1, t_2 \dots t_n, s_1, s_2 \dots s_n; t_1', t_2' \dots t_n', s_1', s_2' \dots s_n')$ $= e^{-\frac{i}{\hbar}(s_1 t_1' + s_2 t_2' + \dots s_n t_n')}$

but that to within a certain natural equivalence σ is the only possible "non degenerate" multiplier for R^{2n}. Thus Weyl demonstrated that one is led naturally to self adjoint operators satisfying the Heisenberg commutation relations if one simply considers the most general projective unitary representation of the m parameter commutative Lie group R^{2n}. (One can show that no non degenerate multipliers σ exist for R^{2n+1}). He attached great significance to this fact and to the fact (of which he gave a heuristic proof) that to within unitary equivalence there is only one projective unitary representation of R^{2n} which is irreducible and has non degenerate multiplier σ. Correspondingly of course there is to within unitary equivalence only one irreducible $2n$ tuple of self adjoint operators satisfying the Heisenberg commutation rules. As Weyl expressed it "The kinematical structure of a physical system is expressed by an irreducible group of unitary ray representations in system space".

In close connection with the above Weyl showed that the smallest and simplest commutative group with a non degenerate projective multiplier σ is the four element group $z_2 \times z_2$. Once again there is an essentially unique projective representation. It is two dimensional and the four two by two matrices concerned may be taken as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and the three matrices $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ now familiar as the Pauli spin matrices.

These remarks constitute Weyl's contribution to the first of the two problems stated in the introduction to his paper. While more suggestive than persuasive or logically compelling, they are of importance as perhaps the first step in the program of deriving fundamental relationships in quantum mechanics from group theoretical symmetry principles in a program which I feel it appropriate to call Weyl's program and to distinguish fairly sharply from the related important program inaugurated and much developed by Wigner. (Later of course each made contributions to the others programs).

As suggested in the introduction to this account of Weyl's contributions and as will be explained below Weyl's uniqueness theorem, as rigorized by Stone and von Neumann in 1930, admits a sequence of natural generalizations; the last of which may be used to give a much more logically compelling deduction of the Heisenberg commutation relations — and quite a bit more besides. These developments show the fundamental soundness of Weyl's intuition as expressed in his semi mystical answer to problem 1.

As far as the third and final section of Weyl's paper is concerned we shall mention only his emphasis on the point that when one integrates the Schrödinger equation $\frac{d\phi}{dt} = iH\phi$ one obtains $\phi_t = e^{iHt}\phi_0$ so that unitary representations of the real line enter once again: The change of a state with time is explicitly described by the action of such a representation.

Last but not least we come to Weyl's celebrated book "Gruppentheorie und Quantenmechanik" published in 1928 and based on a course of lectures given at the E.T.H. in Zurich during the winter semester 1927-28. Of the five chapters of this book I and III largely consist of mathematical preliminaries. The first contains an exposition of the theory of (mainly) finite dimensional Hilbert spaces and the third an exposition of the unitary representation theory of finite groups and compact Lie groups. The heart of the book lies in Chapters II, IV and V. Chapter II contains one of the earliest systematic coherent accounts of quantum mechanics as a whole. Perhaps only Dirac had as complete an overall view earlier, but his early accounts are less complete and well organized. The section of this chapter dealing with the notion of a particle in an electromagnetic field contains the following words (quoted from the English translation of the second edition "The field equations for the potentials ψ and ϕ of the material and electromagnetic waves are invariant under the simultaneous replacement of ψ by $e^{i\lambda}\psi$ and Φ_α by $\Phi_\alpha - \frac{h}{e}\frac{\partial\lambda}{\partial x_\alpha}$; here λ is an arbitrary function of the space-time coordinates. This "principle of gauge invariance" is quite analogous to that previously set up by the author on speculative grounds, in order to arrive at a unified theory of gravitation and electricity. But I now believe that this gauge invariance does not tie together electricity and gravitation but rather electricity and matter in the manner described above." This enunciation of the principle of gauge invariance is again a remarkable anticipation of future work. A quarter of a century later it was generalized by Yang and Mills in a now famous paper and in its generalized form has revolutionized elementary particle physics since the middle 1960's.

Chapter IV entitled "Application of the theory of groups to quantum mechanics" is divided into four parts. Part A, subtitled "the rotation group" is a complete and detailed exposition of how the unitary representation theory of the rotation group explains, organizes and illuminates the theory of atomic spectra. This is the application mentioned above which was discovered and worked out by Wigner and von Neumann. Part B, "the Lorentz group is based on Dirac's celebrated paper of 1928 presenting a relativisticaly invariant quantum mechanical theory of the electron. However it is much more than a simple exposition of Dirac's work. Weyl presents the material in a different way and discusses its significance and implications in depth. Part C "the permutation group" is concerned with the implications for the quantum mechanics of n interacting identical particles of the natural action of S_n the permutation group on n things on the tensor product

$\mathcal{H} \times \mathcal{H} \cdots \times \mathcal{H}$ of n copies of the single particle Hilbert space. For each permutation π in S_n there is a unique unitary operator W_π which maps the vector $\phi_1 \times \phi_2 \cdots \times \phi_n$ in $\mathcal{H} \times \mathcal{H} \cdots \times \mathcal{H}$ into $\phi_{\pi(1)} \times \phi_{\pi(2)} \cdots \times \phi_{\pi(n)}$ and the mapping $\pi \rightarrow W_\pi$ is a unitary representation of S_n. When one decomposes this representation as a direct sum of multiples of the various irreducible representations of S_n one obtains a direct sum decomposition of H x H .... x H, two components which play a special role in quantum mechanics. These are the components defined by the two one dimensional representations the identity representation I and the representation J which takes every "odd" permutation into −1 and every even representation into 1. One calls the subspace corresponding to I the symmetric subspace and that corresponding to J the anti symmetric subspace. Moreover one refers to these subspaces as the symmetrized and anti symmetrized tensor products respectively. It is a fact of great importance that the Hilbert space of states for a system of n identical particles (with one particle space H) is not H x H .... x H as one might be inclined to suppose but either the symmetric or the anti symmetric subspace. Since either case may occur one has a fundamental division of all particles into two categories. Nowadays one speaks of them as bosons and fermions respectively. Weyl discusses how this circumstance implies that interchanging two identical particles makes no physical difference whatever, how the fact that electrons are fermions implies the Pauli exclusion principle, how the latter together with the fact that electrons have spin $\frac{1}{2}$ explains the periodic table and how "quantizing a field" leads to particles (field quanta) which are bosons or fermions according as one uses the Heisenberg commutation relations or the anti commutation relations of Jordan and Wigner. The final part D "quantum kinematics" is an exposition of part II of the paper of Weyl discussed at length above.

The final Chapter V is widely considered to be the most difficult part of the book. Its starting point is Wigner's observation about the utility of the representation theory of the permutation group on the analysis of atomic spectra. However Weyl carries the work much further and applies it to the structure of molecules and the elucidation of the chemical notion of valence. This program requires a considerable purely mathematical development and around seventy percent of the chapter can be read as pure mathematics. It will be easier to explain more fully after we have explained the notions of "induced representation" and "system of imprimitivity" which arise naturally when one generalizes the Stone-von Neumann rigorization of Weyl's theorem on the uniqueness of the irreducible solutions of the Heisenberg commutation relations.

Considering the comprehensiveness of Weyl's book, the early date at which it was written and the wealth of original ideas which it contains one cannot fail to be tremendously impressed by Weyl's achievement or to understand why it is considered one of the great classics of mathematical physics.

We turn our attention now to an account of how later developments inspired directly or indirectly by Weyl's paper of 1927 "Quanten mechanik und Gruppentheorie" led ultimately to a considerable improvement of Weyl's answer to problem I. It is interesting (and instructive) to note that the intermediate stages as well as the final result seem far removed from physics and that the final result has other applications some of which are also important for quantum mechanics.

The first step came in 1930 with the publication of a celebrated short note by M.H. Stone entitled "Linear transformations in Hilbert space III. Operational methods and group theory". In this note Stone announces two mathematical theorems with a sketch of their proofs and states that an account of their significance for physics will be found in the above cited paper of Weyl. One of them is a theorem about arbitrary continuous unitary representations of the additive group of the real line R and bears the same relationship to the decomposability theorem for finite dimensional group representations as Hilbert's spectral theorem bears to the diagonizability theorem for finite dimensional self adjoint matrices. Just as in the spectral theorem one associates to each such representation $t \to V_t$ a unique projection valued measure $E \to P_E$ defined on R. However here the relationship $(H(\varphi), \psi) = \int_{-\infty}^{\infty} \lambda \, d(P_\lambda(\varphi), \psi)$ is replaced by $(V_t(\varphi), \psi) = \int_{-\infty}^{\infty} e^{it\lambda} d(P_\lambda(\varphi), \psi)$ the equation holding for all real t. The theorem goes on to state that the representation V and the projection valued measure P determine one another uniquely and that every P occurs. It will be convenient to refer to this theorem as the spectral theorem for unitary representations of R.

Consider now the two one to one correspondences set up by the spectral theorems for self adjoint operators and continuous unitary representations of R respectively. Having a common term (the projection valued measures on R) they define a one-to-one correspondence between self adjoint operators H and continuous unitary representations $t \to V_t$ of R. It is not difficult to check that H and V correspond in this way if and only if $V_t = e^{itH}$ for all t. The fact that every V is so related to some unique H is usually referred to as "Stone's theorem". The other theorem stated by Stone is simply the uniqueness theorem for operators satisfying the Heisenberg commutation relations in the form involving group representations given it three years earlier by Weyl. Stone did not publish his proof and the first published proof is due to von Neumann. One speaks of the Stone von Neumann theorem.

The second step came in 1933 when A. Haar proved that every separable locally compact group admits a measure which is invariant under right (left) translation. That this measure is essentially unique was proved slightly later by von Neumann. The significance of this result is that it paved the way for extending the theory of unitary group representations from compact Lie groups to arbitrary compact topological groups and on to topological groups which are not even compact provided that they are locally compact in the sense that every point has a compact neighbourhood. Haar himself observed that using his existence theorem one could extend the Peter Weyl theorem to all compact topological groups.

The third step came in 1934 when L. Pontryagin and E.R. van Kampen developed their celebrated duality theorem for locally compact commutative groups. Let G be a locally compact commutative group. Let $\hat{G}$ denote the set of all continuous characters of G i.e. the group of all continuous functions χ from G to the complex numbers of modulus one such that $\chi(xy) = \chi(x)\chi(y)$. The product of two continuous characters is evidently again such and under this operation $\hat{G}$ is again a commutative group. It is even a locally compact topological group with respect to a topology which may be loosely be described as the topology of uniform convergence on compact subsets. One calls it the character group of G or the dual group of G. Of course one may now form $\hat{\hat{G}}$ the dual of the dual and ask about its relationship to G. It is immediate that we may almost think of G as contained in $\hat{\hat{G}}$. Indeed for each $x \in G$ the function $\chi \to \chi(x)$ on $\hat{G}$ is in fact a continuous character on $\hat{G}$ and hence a member of $\hat{\hat{G}}$ which we may denote by f_x. The mapping $x \to f_x$ is clearly multiplication preserving and if it happens to be one-to-one we have an isomorphism of G onto a subgroup of $\hat{\hat{G}}$. The duality theorem originated by Pontryagin and completed on various points by van Kampen asserts that $x \to f_x$ is always one to one that the subgroup of $\hat{\hat{G}}$ onto which it maps G on the <u>whole</u> of $\hat{\hat{G}}$ and that this isomorphism of G with $\hat{\hat{G}}$ is an isomorphism of topological groups. In other words locally compact commutative groups occur in dual pairs; each member of any pair being the dual of the other. In some cases G and $\hat{G}$ may be isomorphic as topological groups so that G is self dual; for example this is so when G is finite and when G is the additive group of a finite dimensional real vector space. In general however this is not so. In particular $\hat{G}$ is compact whenever G is discrete and vice versa.

The fourth step came a decade later in 1944. Stone's formulation and proof of the spectral theorem for continuous unitary representations of R was, in effect, an extension of the theory of group representations to one particular non compact group. Of course R is locally compact as well as commutative and it is natural to wonder if one can go further and extend this theory to all locally compact commutative groups. One can and this was realized independently at about the same time by Ambrose in the U.S.A., Godement in France, and Naimark in the Soviet Union. All of these mathematicians published their work in 1944. To see how to generalize the formula $\int_{-\infty}^{\infty} e^{it\lambda}\, d(P_\lambda(\varphi)\cdot\psi) = (V_t(\varphi)\cdot\psi)$ one has only to realize that for each real λ the function $t \to e^{i\lambda t}$ is a character χ'_λ of R and that $\lambda \to \chi'_\lambda$ is an isomorphism of R with $\hat{R}$. Using this isomorphism, $E \to P_E$ can be thought of as a projection valued measure on $\hat{R}$ and one can rewrite Stone's formula as

$$(V_t(\varphi)\cdot\psi) = \int_{\chi'\in\hat{R}} \chi'(t)\, d(P_{\chi'}(\varphi)\cdot\psi)$$

This formula of course makes equal sense when R is replaced by any locally compact commutative group G and $\hat{R}$ by $\hat{G}$ the Pontryagin dual of G. The theorem of Ambrose, Godement and Naimark now simply asserts the existence of a one to one correspondence between all continuous unitary representations V of G and all projection valued measures $E \to P_E$ on the dual $\hat{G}$ such that $(V_t(\varphi)\cdot\psi) = \int \chi'(t)\, d(P_{\chi'}(\varphi)\cdot\psi)$ for all $t \in G$ and all ϕ and ψ in H(V) the Hilbert space of V.

The fifth step came in 1949 when the present author published a paper in the Duke Mathematical Journal entitled "On a theorem of Stone and von Neumann". It was based on the following sequence of observations.

(1) The factor $e^{\frac{i}{h}(s_1t_1 + \cdots s_nt_n)}$ which occurs in the Weyl form of the Heisenberg commutation relation can be interpreted as $\chi'(t)$ where $t = t_1 t_2 \ldots t_n$ is an element of the group of all n tuples of real numbers under addition and χ' is the character $t_1, t_2 \ldots t_n \to e^{\frac{i}{h}(s_1 t_1 + \cdots s_n t_n)}$

(2) With this interpretation the Heisenberg commutation relations have an obvious generalization which may be written down for a pair U, V where U is a continuous unitary representation of an arbitrary locally compact commutative G and V is a continuous unitary representation of the dual group $\hat{G}$. This generalization reads:

$U_x V_{\chi} = \chi(x) V_{\chi} U_x$ for all $x \in G$ and all $\chi \in \hat{G}$.

(3) Using the generalized spectral theorem of Ambrose, Godement and Naimark one has a projection valued measure on $G = \hat{\hat{G}}$ canonically associated to V and an elementary argument shows that V and U satisfy the commutation relation $U_x V_{\chi} = \chi(x) V_{\chi} U_x$ if and only if U and P satisfy the commutation relation

$$U_x P_E = P_{[E]x^{-1}} U_x$$

for all x and E where $[E]x^{-1}$ denotes the translate of E by x^{-1}

(4) Because of (3) studying pairs U, V which satisfy the generalized commutation relation written down under (2) can be reduced to studying pairs, U, P which satisfy the commutation relation $U_x P_E = P_{[E]x^{-1}} U_x$ written down under (3). However both U and P are defined on the same group G; no reference whatever being given to the dual $\hat{G}$ of G. Moreover this new form of the commutation relation makes sense even when G is non commutative. This raises the question of proving a generalization of the Weyl-Stone-von Neumann uniqueness theorem that applies not only to dual pairs of locally compact commutative groups but to arbitrary locally compact groups — commutative or not. This can be done and the generalized theorem is the main result of the Duke Journal paper cited above. In this paper the group G is assumed to have a countable basis for the open sets but this restriction was removed by Loomis in 1952.

It is perhaps worth stating the theorem explicitly in such a way as to present a concrete example of a pair satisfying the commutation relation. Let G be a locally compact group and let μ be a right invariant Haar measure for G. Form the Hilbert space $\mathcal{L}^2(G,\mu)$ of all square summable complex valued functions on G. For each x in G let U^0_x denote the unitary operator which takes f into g where g(y) = f(yx). Then $x \to U^0_x$ is a continuous unitary representation of G known as the regular representation. Next for each Borel subset E of G let P^0_E denote the projection operator which takes f into g where g is the function which agrees with f on E and is zero outside of E. One verifies that $E \to P^0_E$ is a projection valued measure on G and a straightforward computation shows that

$$U^0_x P^0_E = P^0_{[E]x^{-1}} U^0_x \qquad *$$

for all x and E. Finally one can prove that the pair U, P^0 is irreducible in the sense that no proper closed subspace of $\mathcal{L}^2(G,\mu)$ is invariant under all U^0_x and all P^0_E. Now let U be any unitary representation of G and let P be any projection valued measure on G such that the P_E and the U_x operate in the same Hilbert space H. Suppose that U and P satisfy $*$ and are jointly irreducible. Our generalized uniqueness theorem then asserts that there exists a unitary operator W mapping H onto $\mathcal{L}^2(G,\mu)$ so that $WP_EW^{-1} = P^0_E$ and $W U_x W^{-1} = U^0_x$ for all x and E.

Shortly after this paper was submitted the author noticed that a still further generalization was conceivable. One can replace the projection valued measure $E \to P_E$ on G by a projection valued measure $E \to P_E$ defined on some space S on which G acts as a transformation group. The commutation relation $U_x P_E = P_{[E]x^{-1}} U_x$ still makes sense provided that we interpret $[s]x$ as the transform of s in S by x in G. Moreover it reduces to the one discussed above when S=G and $[s]x = sx$ is just group multiplication. The question that now presents itself is the following. Is the uniqueness theorem stated above still true at this new level of generality? The answer is no. However in an important special case one can analyze the non uniqueness completely and so produce a theorem that generalizes the uniqueness theorem stated above. More specifically one can find all possible solutions of the commutation relation in question. The special case in which this can be done is that in which S is the "homogeneous space" G/K defined by some closed subgroup K of G; that is the space whose elements are the right K cosets Kx and the transform of Kx by y is the right coset $K(xy)$. It is evident that the action of G on G/K is "transitive" in the sense that given Kx and Ky in G/K there exists z in G such that $[Kx]z = Ky$. We need only choose $z=x^{-1}y$. Conversely every transitive G space S with suitable regularity properties is isomorphic to a coset space G/K. When S = G/K what one finds instead of uniqueness is that the possible solutions of the commutation relation above have equivalence classes that correspond one to one in a natural way to the equivalence classes of unitary representations of K. Moreover a solution of the commutation relations is irreducible if and only if the corresponding unitary representation of K is irreducible. Of course when K = $\{e\}$ so that S = G there is only one irreducible representation of K. That is why there is uniqueness when S = G.

To gain some insight into the situation and a new point of view regarding the meaning of the commutation relation it is useful to consider the special case in which there are only a discrete countable infinity of right K cosets so that S is a countable discrete set of points. In that case the projection valued measure $E \rightarrow P_E$ is completely determined by the projections $P_{\{s\}}$ assigned to the one point subsets $\{s\}$ of S. Moreover the ranges of these projections constitute a direct sum decomposition of the underlying Hilbert space H.

$H = \sum_{s \in S} H_s$ where H_s is the range of $P_{\{s\}}$. A very simple computation now shows that the commutation relation $U_x P_E = P_{[E]x^{-1}} U_x$ holds if and only if each U_x carries the subspace H_s onto the subspace $H_{[s]x^{-1}}$

Thus the operators U_x while they do not leave the subspace s H_s invariant in general, they do preserve their identity; merely permuting them among themselves. Now consider the special case in which $s = K$, the right coset s_0 containing the identity. Then for $x \in K$, $U_x(\mathcal{H}_{s_0}) = H_{s_0}$ so that $k \rightarrow U_k$ defines a unitary representation L of the subgroup K by operators in the subspace H_{s_0}. A little reflection should convince the reader that once one knows K and the unitary representation L of K the pair U, P can be reconstructed and is uniquely determined by K and L. Thus finding all unitary equivalence classes of pairs U, P satisfying the commutation relations reduces, in this case at least, to finding all unitary equivalance classes of continuous unitary representations of the subgroup K.

Quite generally if U is a unitary representation of a group G and $H(U) = H_1 + H_2 \oplus \ldots$ is a direct sum decomposition of the underlying Hilbert space $H(U)$ one refers to this decomposition as a "system of imprimitivity" for the representation U provided that each U_x simply permutes the subspaces H_j among themselves. If in particular, for each pair i and j, there is an x such that $U_x(H_i) = H_j$ one refers to a transitive system of imprimitivity. This terminology is suggested by a closely related notion in the theory of permutation groups. The fact that a unitary representation of a finite group G together with a transitive system of imprimitivity for it is determined by a unitary representation L of a subgroup of G was already known to Frobenius. We may now think of Frobenius's result as finding the most general solution of our last generalization of the Heisenberg commutation relations in the very special case of finite G and S. The fact that Frobenius's result has a more or less complete generalization to the case in which G is a general separable locally compact group and K is an arbitrary closed subgroup is the content of a paper published by the present author in 1949 and entitled "Imprimitivity for representations of localy compact groups I". It will be useful to refer to the main result of this paper as the "imprimitivity theorem". In its replacement of a discrete system of imprimitivity by a projection valued measure it is strongely reminiscent of the passage from the classical diagonalization theorem for matrices to Hilbert's spectral theorem.

On the face of it the imprimitivity theorem seems to have left its original inspiration far behind and to have very little connection with physics. However as we shall now indicate and as was promised earlier it is just what is needed to give a more satisfying answer to Weyl's problem I. Let S denote physical space and let $\mathcal{E}$ denote the group of all isometries of S. Then in the usual Euclidean model for space (and in certain other models as well) $\mathcal{E}$ acts on S as a <u>transitive</u> transformation group. Consider now the quantum mechanical model of a single free particle. Let H be the Hilbert space of pure states in the von Neumann formulation. The position coordinates and the velocity or momentum components will then be associated with certain self adjoint operators in H. Which operators? That is Weyl's problem I. To answer it we begin with the observation that the position coordinate observables may be discussed in a coordinate free way by replacing them with two valued observables which take on the value one or zero according as the particle is observed to be in a certain region E of space or not. Each such two valued observable will necessarily (according to the von Neumann scheme) be associated with a projection operator P_E and it is easy to argue that $E \to P_E$ must satisfy the conditions defining a projection valued measure on S. Once P is known it is easy to deduce the projection valued measure associated with <u>any</u> real valued coordinate i.e. any real valued Borel function f on S. It is just the projection valued measure $E \to P_{f^{-1}(E)}$ on the real line R. In these terms Weyl's problem I becomes (in part) What projection valued measure P on S has nature chosen (or must nature choose). The answer is based on the hypothesis that nature's choice will reflect the symmetry of space as reflected in the action of the group $\mathcal{E}$ on S; that the laws of nature must be independent of position and orientation in space. It is not hard to argue that this principle implies the existence for each γ in $\mathcal{E}$ of a certain unitary operator U_γ which describes the transformation of the states associated with a rigid motion of space and that the mapping $\gamma \to U_\gamma$ is a projective unitary representation associated with some multiplier σ. Of course U and P cannot be independently chosen they must be so related that rigid motions change the position observables in the appropriate way. Analysis of this leads to the conclusion that U and P must be so related that $U_\gamma P_E U_\gamma^{-1}$ is just $P_{[E]\gamma^{-1}}$ But this relation is equivalent to our generalized commutation relation $U_\gamma P_E = P_{[E]\gamma^{-1}} U_\gamma$ and we may apply the imprimitivity theorem (which is valid also for projective representations). The conclusion is that, up to unitary equivalence there is just one possible pair P, U for each projective unitary representation L of the subgroup K of $\mathcal{E}$ leaving fixed an "origin" s_0 in space. K is of course just the compact group of all rotations about a fixed point and it is well known that it has precisely one irreducible unitary projective representation of every positive integer dimension. Thus one has an overall view of all possible pairs P, U and one sees in particular that there are only a discrete countable set of such. Given P, U one not only knows the operators corresponding to the coordinate observables but also the operators corresponding to the linear and angular momentum observables. The latter are derived from U by the general principle valid in both classical and quantum mechanics relating integrals of the motion to one parameter symmetry groups. In the special case in which the representation L of K is one dimensional one is led to the classical form for the operator corresponding to position and momentum observables for a particle without spin. More generally if L is the projective representation of dimension $j = 1,2,3,\cdots$ one is led to the classical form for the operators corresponding to the position and momentum observables for a particle of spin $\frac{j-1}{2}$. In particular one is led automatically to the Pauli matrices for particles of spin $\frac{1}{2}$. For further details including the extension to interacting particles and references to related work of Wigner and of Wightman the author is referred to the middle sections of the author's book "Unitary group representations in physics, probability and number theory" W.A. Benjamin 1978.

We conclude with some very brief indications concerning connections of the above with the fifth chapter of Weyl's book. First of all given any unitary representation L of any closed subgroup K of any separable locally compact group G there always exists a pair P, U for the action of G on G/K such that the defining representation of K is L. The representation U is uniquely determined by L and is known as the unitary representation of G <u>induced</u> by L. It is convenient to denote this induced representation by the symbol U^L. One finds that many interesting locally compact groups G have most if not all their irreducible representation of the form U^L where L is a lower dimensional representation of a proper subgroup. Moreover the imprimitivity theorem is a useful tool in proving such things. Indeed one can often detect a <u>transitive</u> system of imprimitivity for the representation in question and this implies that the representation is induced.

In his fifth chapter Hermann Weyl is concerned with the tensor product of n replicas of the same Hilbert space H. This Hilbert space is the space of states of a single particle moving in a potential field and is also the space H(V) of a unitary representation V of a compact group K. As explained earlier in this paper there is a natural unitary representation W of the symmetric group S_n in $H \otimes H \ldots \times H$. Now $V \times V \ldots \times V$ is a representation of the product group $K \times K \times K \ldots \times K$ where in each case there are n factors and W and $V \times V \ldots \times V$ both act in the same product Hilbert space $H \times H \ldots \times H$. These two representations combine to define a representation $V^n W$ of a certain "twisted" product of the two groups $K \times K \ldots \times K$ and S_n. Here we define the product of $x_1, x_2 \ldots x_n, \pi_1$ and $y_1, y_2 \ldots y_n \pi_2$ to be $x_1 \pi_1(y_1), x_2 \pi_1(y_2), \ldots x_n \pi_1(y_n))$
One is interested in determing the structure of the restriction of $V^n W$ to the subgroup of the twisted product consisting of all $x_1, x_2 \ldots x_n, \pi$ with $x_1 = x_2 \ldots = x_n$. This subgroup is of course isomorphic to $K \times S_n$. Consider now a decomposition of V into irreducibles $H(V) = H_1 + H_2 + \ldots$ where each H_j is an invariant subspace. Then each product space $H_{j_1} \times H_{j_2} \ldots \times H_{j_n}$ will be a subspace of $H \times H \ldots \times H$ and all of these

subspaces together constitute a direct sum decomposition of $H \times H \,..\, \times H$ These subspaces are of course invariant under the representation V^n of $K \times K \ldots \times K$ but not under the representation $V^n W$ of the twisted product of $K \times K \ldots \times K$ with S_n. However, and here is the key point, they do constitute a discrete system of imprimitivity for $V^n W$. It is not a transitive system but one may decompose it into transitive pieces and thus decompose $V^n W$ as a discrete direct sum of induced representations. The general theory of induced representations tells one how to study the restrictions of these induced representations to subgroups and in particular to $\tilde{K}^n \times S_n$ where $\tilde{K}^n$ is the "diagonal" subgroup of $K \times K \ldots \times K$ mentioned above.

Weyl does not explain what he is doing in these terms. However if one does so and is familiar with the theory of induced representations one finds oneself led automatically to many of the main arguments. The writer hopes some day to explain all this in detail in an article entitled "Weyl's fifth chapter revisited".